SELF-CONSISTENT FIELD APPROXIMATION FOR AN ELASTIC COMPOSITE MEDIUM

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The self-consistent field method enables one to obtain complete and consistent solutions when solving the problem of determining pertinent elastic properties of inhomogeneous or composite media even if they involve strong inhomogeneities and the admixtures are highly concentrated. The latter, together with the simplicity and physical clarity of the basic assumptions for the self-consistent field, has attracted the attention of many researchers [1-3]. In [4-6] the concept of self-consistency was implicitly employed when solving problems for a medium containing fields of ellipsoidal inhomogeneities. In these articles the self-consistent field method was employed to analyze a composite medium containing isotropic random fields of defects (macroisotropic medium). The self-consistent field method can in actual fact be also applied to the correlated or even to the regular structures [7, 8]. A serious obstacle in employing self-consistent solutions is the lack of any estimates for the error of a given method when systems of the elastic composite medium type are investigated. Thus, a direct comparison of self-consistent solutions with the exact ones is of considerable value as a criterion of applicability of the self-consistent field method in the mechanics of composite media. In the present article a composite medium containing an arbitrary random field of ellipsoidal inhomogeneities is investigated. For the isotropic admixture field and for the case in which the admixtures form a regular lattice the tensors of pertinent elastic properties of the medium are constructed. The results of numerical evaluations for a cubic lattice of isotropic spherical admixtures in an isotropic medium are cited. An analysis of the plane problem enables one to compare the solution obtained by the self-consistent field method with the exact solutions obtained in [9] for regular admixture lattices.

§1. An infinite elastic homogenous medium (the base medium) is considered whose properties form the tensor of the elasticity moduli $\mathbf{L}$; the latter contains a homogenous random field of ellipsoidal inclusions with the elasticity moduli $\mathbf{L}+\mathbf{L}_1$, where $\mathbf{L}_1$ is a random tensor remaining constant within each inclusion.

It is known (see, for example, [6, 10], that the deformation tensor ε in a medium with inhomogeneities and with applied external field ε_0 satisfies the equation

$$\boldsymbol{\varepsilon}(\mathbf{r}) + \int \mathbf{K}(\mathbf{R}) \cdot \mathbf{L}_1(\mathbf{r}') \cdot \boldsymbol{\varepsilon}(\mathbf{r}') \Theta(\mathbf{r}')\, dV' = \boldsymbol{\varepsilon}_0, \quad \mathbf{R} = \mathbf{r} - \mathbf{r}', \tag{1.1}$$

where $\Theta(\mathbf{r})$ is the characteristic function of the domain V occupied by the inclusions (if s is a point of the medium whose radius-vector is $\mathbf{r}$, then $\Theta(\mathbf{r}) = 1$ for $s \in V$ but $\Theta(\mathbf{r}) = 0$ for $s \overline{\in} V$); dots denote convolution over two indices.

The kernel $\mathbf{K}(\mathbf{R})$ of the integral operator in (1.1) is related to the second derivatives of the Green's function U of the base medium by means of

$$K_{ijkl}(\mathbf{R}) = -[\nabla_k \nabla_l U_{ij}(\mathbf{R})]_{(ik)(jl)} \tag{1.2}$$

(round brackets represent symmetrization over the relevant subscripts).

Some properties of the operator $\mathbf{K}$ are now discussed.

1. On the continuous two-valued tensor functions Φ such that

$$\int_{|\mathbf{R}|>1} \mathbf{K}(\mathbf{R}) \cdot \boldsymbol{\Phi}(\mathbf{r}')\, dV' < \infty,$$

this operator can be defined by the following formula [10, 11]:

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$$\int \mathbf{K}(\mathbf{R}) \cdot \mathbf{\Phi}(\mathbf{r}')\, dV' = \int \mathbf{K}(\mathbf{C}^{-1}\xi - \mathbf{r}) \cdot \mathbf{\Phi}(\mathbf{C}^{-1}\xi) \det \mathbf{C}^{-1} dV_\xi + \mathbf{A} \cdot \mathbf{\Phi}(\mathbf{r}), \quad \xi = \mathbf{C}\mathbf{r}', \tag{1.3}$$

where $\mathbf{C}$ is the tensor which determines a nondegenerate affine transformation of the three-dimensional space. The constant tensor $\mathbf{A}$ is equal to

$$\mathbf{A} = \frac{1}{4\pi} \int\limits_{(\Gamma_1)} \widetilde{\mathbf{K}}(\mathbf{C}\mathbf{k})\, d\Gamma, \tag{1.4}$$

where $\widetilde{\mathbf{K}}(\mathbf{k})$ is the Fourier transform of the kernel $\mathbf{K}(\mathbf{R})$; Γ_1 is the area of unit sphere in the k-space. The integral on the right-hand side of (1.3) is understood to be the principal value in the Cauchy sense.

2. The operation of the operator $\mathbf{K}$ on the constant tensors is expressed by an integral which diverges at zero and at infinity. It can be shown that this integral cannot be uniquely regularized, its value being defined by the sense it has in the problem under consideration.

Let $\mathbf{L}_1$ in (1.1) be a constant tensor and let also the stress field in the medium be constant and equal to the external field $\sigma_0 = \mathbf{L} \cdot \varepsilon_0$. It is obvious that in this case the deformation tensor of the medium is

$$\varepsilon = (\mathbf{L} + \mathbf{L}_1)^{-1} \cdot \sigma_0.$$

On the other hand, the solution of Eq. (1.1) has this particular form if the regularization of the integral expressing the operation of the operator $\mathbf{K}$ over the constant two-valued tensors $\mathbf{\Phi}_0$ is defined by the relation

$$\int \mathbf{K}(\mathbf{R}) \cdot \mathbf{\Phi}_0 dV = \mathbf{G} \cdot \mathbf{\Phi}_0, \mathbf{G} = \mathbf{L}^{-1}. \tag{1.5}$$

Such regularization in the above-shown meaning will be used by us later on.

For the deformation field ε inside the inclusions (in the domain V) we have an equation which is a corollary of (1.1):

$$\Theta(\mathbf{r})\varepsilon(\mathbf{r}) + \int \mathbf{K}(\mathbf{R}) \cdot \mathbf{L}_1(\mathbf{r}') \cdot \varepsilon(\mathbf{r}')\, \Theta(\mathbf{r}')\, \Theta(\mathbf{r})\, dV' = \Theta(\mathbf{r})\, \varepsilon_0. \tag{1.6}$$

It is noticed that the solution is continued in a unique manner into the domain $\bar{V}$ (a complement of V with respect to the entire space) if ε is known inside V; this follows directly from (1.1).

§2. Now let the external field be homogenous (constant).

The suppositions of the self-consistent field method when applied to the problem under consideration can be formulated in the following way: 1) each insertion of an actual realization of a random field of inhomogeneities is considered as a separate ellipsoidal inclusion into the basic medium; 2) the deformation field $\hat{\varepsilon}$ in which all inclusions are found consists of the external field ε_0 and of the field induced by the surrounding inhomogeneities. It is assumed that this field is the same for all inclusions; an approximation is to be made available for the equivalent field $\hat{\varepsilon}$. It is assumed that the field $\hat{\varepsilon}$ is constant. This is valid within the volume occupied by a typical inclusion, the total field due to all the surrounding inhomogeneities varying only slightly.*

If the field $\hat{\varepsilon}$ is constant, then within the framework of the self-consistent field method the deformation field inside each inclusion (the latter following from the results of [10]) is of the form

$$\varepsilon = (\mathbf{I} + \mathbf{A} \cdot \mathbf{L}_1)^{-1} \cdot \hat{\varepsilon}, \tag{2.1}$$

where $\mathbf{I}$ is the identity four-valued tensor. The tensor $\mathbf{A}$ is of the form (1.4), and the tensor $\mathbf{C}$, which appears in the definition of $\mathbf{A}$, determines an affine transformation which transforms the ellipsoid domain occupied by the inclusion into a ball.

An equation for the equivalent field $\hat{\varepsilon}$ is obtained by substituting (2.1) into (1.6) and by averaging the result over the ensemble of inclusion fields:

$$\{\langle \Theta(\mathbf{r})\, [\mathbf{I} + \mathbf{A}(\mathbf{r}) \cdot \mathbf{L}_1(\mathbf{r})]^{-1} \rangle + \int \mathbf{K}(\mathbf{R}) \cdot \langle \mathbf{L}_1(\mathbf{r}') \cdot [\mathbf{I} + \mathbf{A}(\mathbf{r}') \cdot \mathbf{L}_1(\mathbf{r}')]^{-1}\, \Theta(\mathbf{r}')\, \Theta(\mathbf{r}) \rangle\, dV' \} \cdot \hat{\varepsilon} = p\varepsilon_0, \tag{2.2}$$

where $p = \langle \Theta(\mathbf{r}) \rangle$ is the concentration of inclusions.

Let us now consider the average appearing under the integral sign in (2.2):

$$\langle \mathbf{L}_1(\mathbf{r}') \cdot [\mathbf{I} + \mathbf{A}(\mathbf{r}') \cdot \mathbf{L}_1(\mathbf{r}')]^{-1} \Theta(\mathbf{r}')\Theta(\mathbf{r}) \rangle.$$

*This form of the self-consistent field method differs from the one adopted, for example, in [2] in that each inclusion is regarded as isolated in a medium with elastic properties equal to the pertinent properties of a composite medium, the field in which any inclusion is located being assumed equal to the external field ε_0.

When averaging over the ensemble of inclusion fields the contribution toward this average is only given by those realizations for which $\Theta(\mathbf{r})\Theta(\mathbf{r}')=1$, that is, the points $\mathbf{r}$ and $\mathbf{r}'$ must simultaneously enter the domain V occupied by the inclusions. The sought average for homogenous inclusion fields can be represented by the sum

$$\langle \mathbf{L}_1(\mathbf{r}')\cdot[\mathbf{I}+\mathbf{A}(\mathbf{r}')\cdot\mathbf{L}_1(\mathbf{r}')]^{-1}\Theta(\mathbf{r}')\Theta(\mathbf{r})\rangle = \mathbf{\Psi}_0(\mathbf{R})+\mathbf{\Psi}(\mathbf{R}), \tag{2.3}$$

where $\mathbf{\Psi}_0(\mathbf{R})$ is a part which is related to the points $\mathbf{r}$ and $\mathbf{r}'$ finding themselves in the same inclusions, whereas $\mathbf{\Psi}(\mathbf{R})$ is a part related to the points finding themselves in different inclusions.

An elementary geometric analysis enables one to represent the function $\mathbf{\Psi}_0(\mathbf{R})$ in the form

$$\mathbf{\Psi}_0(\mathbf{R}) = \langle \mathbf{L}_1\cdot[\mathbf{I}+\mathbf{A}\cdot\mathbf{L}_1]^{-1}(1/V_0)J(\mathbf{R}, c_1, c_2, c_3, \mathbf{Q})\rangle, \tag{2.4}$$

where $J(\mathbf{R}, c_1, c_2, c_3, \mathbf{Q})$ is the volume inside the ellipsoidal domain occupied by an inclusion with the semi-axes c_1, c_2, c_3 (the orientation in space of an inclusion is determined by an orthogonal random tensor $\mathbf{Q}$) and if the point $\mathbf{r}'$ finds itself in this domain, then the point $\mathbf{r}$ must also be there. The averaging in (2.4) is carried out over the ensemble of distribution functions of the inclusion dimensions and their properties; V_0 is the mean volume per one inclusion. By affine transformation $\mathbf{C}(\boldsymbol{\xi}=\mathbf{CR})$, which transforms the ellipsoid into a unit ball, the function $J(\mathbf{R}, c_1, c_2, c_3, \mathbf{Q})$ is mapped into the spherically symmetrical function

$$J(\mathbf{C}^{-1}\boldsymbol{\xi}) = J'(|\boldsymbol{\xi}|).$$

It is noticed, moreover, that

$$J(0, c_1, c_2, c_3, \mathbf{Q}) = v_c \text{ and } J\to 0 \text{ as } |\mathbf{R}|\to\infty, \tag{2.5}$$

where $v_c=(4/3)\pi c_1c_2c_3$ is the volume of the ellipsoid with the semiaxes c_1, c_2, c_3.

The integral

$$\int \mathbf{K}(\mathbf{R})\cdot\Psi_0(\mathbf{R})\,dV = \left\langle \frac{1}{V_0}\int \mathbf{K}(\mathbf{C}^{-1}\boldsymbol{\xi})\cdot\mathbf{L}_1\cdot(\mathbf{I}+\mathbf{A}\cdot\mathbf{L}_1)^{-1}\times \right.$$
$$\left. \times J'(|\boldsymbol{\xi}|)\det\mathbf{C}^{-1}dV_{\xi}\right\rangle = \left\langle \frac{v_c}{V_0}\mathbf{A}\cdot\mathbf{L}_1\cdot(\mathbf{I}+\mathbf{A}\cdot\mathbf{L}_1)^{-1}\right\rangle \tag{2.6}$$

is now evaluated.

The last equality results from (1.3) if one takes into account that in the case in which $J'(\xi)$ is a spherically symmetrical function the principal value of the integral appearing in (2.6) equals zero. One substitutes (2.3) into (2.2); after some fairly simple transformations one obtains by using (2.5) the equation for the equivalent field $\hat{\boldsymbol{\varepsilon}}$ in the form

$$\left[\mathbf{I}+\frac{1}{p}\int \mathbf{K}(\mathbf{R})\cdot\mathbf{\Psi}(\mathbf{R})\,dV\right]\cdot\hat{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon}_0, \tag{2.7}$$

where the equality

$$\langle \Theta(\mathbf{r})\mathbf{L}_1(\mathbf{r})\cdot[\mathbf{I}+\mathbf{A}(\mathbf{r})\cdot\mathbf{L}_1(\mathbf{r})]^{-1}\rangle = \left\langle \frac{v_c}{V_0}\mathbf{L}_1\cdot(\mathbf{I}+\mathbf{A}\cdot\mathbf{L}_1)^{-1}\right\rangle$$

is used which is valid for homogeneous ergodic inclusion fields. The averaging on the left-hand side is over the ensemble of realizations of the random field, and on the right the averaging is over a random quantity whose distribution function can be determined by the ensemble of distribution functions of the inclusion dimensions and their properties.

It follows from (2.1) and (2.7) that the deformation within any inclusion is determined within the framework of the self-consistent field method by the expression

$$\boldsymbol{\varepsilon} = (\mathbf{I}+\mathbf{A}\cdot\mathbf{L}_1)^{-1}\cdot\left[\mathbf{I}+\frac{1}{p}\int \mathbf{K}(\mathbf{R})\cdot\mathbf{\Psi}(\mathbf{R})\,dV\right]^{-1}\cdot\boldsymbol{\varepsilon}_0. \tag{2.8}$$

Moreover, let the medium be acted upon by an external stress field $\boldsymbol{\sigma}_0$ related to the field $\boldsymbol{\varepsilon}_0$ by the obvious relation

$$\boldsymbol{\varepsilon}_0 = \mathbf{G}\cdot\boldsymbol{\sigma}_0.$$

Then the mean deformation $\langle\boldsymbol{\varepsilon}\rangle$ of the medium with inclusions is obtained by substituting (2.8) into (1.1) and by averaging the result over the realization ensemble:

$$\langle\varepsilon\rangle = \mathbf{G}\cdot\sigma_0 - \int \mathbf{K}(\mathbf{R})\cdot\left\langle \mathbf{L}_1(\mathbf{I}+\mathbf{A}\cdot\mathbf{L}_1)^{-1}\cdot\left[\mathbf{I}+\frac{1}{p}\int \mathbf{K}(\mathbf{R})\times\right.\right.$$

$$\left.\left.\times\,\Psi(\mathbf{R})\,dV\right]^{-1}\Theta(\mathbf{r}')\right\rangle dV'\cdot\mathbf{G}\cdot\sigma_0 = \mathbf{G}\cdot\sigma_0 - \mathbf{G}\cdot\left\langle\frac{v_c}{V_0}\mathbf{L}_1\cdot(\mathbf{I}+\mathbf{A}\cdot\mathbf{L}_1)^{-1}\right\rangle\times$$

$$\times\left[\mathbf{I}+\frac{1}{p}\int \mathbf{K}(\mathbf{R})\cdot\Psi(\mathbf{R})\,dV\right]^{-1}\cdot\mathbf{G}\cdot\sigma_0. \tag{2.9}$$

The last equality follows from (1.5).

We now introduce the tensor of pertinent elastic pliability of a medium with inclusions by means of the relation

$$\langle\varepsilon\rangle = \mathbf{G}_E\cdot\sigma_0.$$

Hence and from (2.9) one obtains

$$\mathbf{G}_E = \mathbf{G} - \mathbf{G}\cdot\left\langle\frac{v_c}{V_0}\mathbf{L}_1\cdot(\mathbf{I}+\mathbf{A}\cdot\mathbf{L}_1)^{-1}\right\rangle\cdot\left[\mathbf{I}+\frac{1}{p}\int \mathbf{K}(\mathbf{R})\cdot\Psi(\mathbf{R})\,dV\right]^{-1}\cdot\mathbf{G}. \tag{2.10}$$

To be able to construct the function $\Psi(\mathbf{R})$ under the integral sign on the right-hand side of the above expression one must have a specific model of the random field of inclusions in the medium. Two extreme cases will be considered: the random field of inclusions is homogenous and isotropic; the inclusions form a regular lattice.

A. For an isotropic field the function $\Psi(\mathbf{R})$ [the expression (2.3)] is spherically symmetrical (it depends only on $|\mathbf{R}|$ and it tends to $p\left\langle\frac{v_c}{V_0}\mathbf{L}_1\cdot(\mathbf{I}+\mathbf{A}\cdot\mathbf{L}_1)^{-1}\right\rangle$ at infinity). Moreover, for all cases the condition $\Psi(0)=0$ is valid, since one excludes the possibility of one point being covered by two different inclusions (the inclusions are not intersecting).

The integral in (2.10) is now evaluated for this case by employing the regularization (1.3), (1.5):

$$\frac{1}{p}\int \mathbf{K}(\mathbf{R})\cdot\Psi(\mathbf{R})\,dV = \frac{1}{p}\int \mathbf{K}(\mathbf{R})\cdot\left[\Psi(\mathbf{R}) - p\left\langle\frac{v_c}{V_0}\mathbf{L}_1\times\right.\right.$$

$$\left.\left.\times(\mathbf{I}+\mathbf{A}\cdot\mathbf{L}_1)^{-1}\right\rangle\right]dV + \mathbf{G}\cdot\left\langle\frac{v_c}{V_0}\mathbf{L}_1\cdot(\mathbf{I}+\mathbf{A}\cdot\mathbf{L}_1)^{-1}\right\rangle =$$

$$= (\mathbf{G}-\mathbf{A}_0)\cdot\left\langle\frac{v_c}{V_0}\mathbf{L}_1\cdot(\mathbf{I}+\mathbf{A}\cdot\mathbf{L}_1)^{-1}\right\rangle. \tag{2.11}$$

By substituting this result in (2.10), one obtains for the tensor of pertinent elastic pliability $\mathbf{G}_E$ and pertinent elastic moduli $\mathbf{L}_E = \mathbf{G}_E^{-1}$ the expressions

$$\begin{aligned}\mathbf{G}_E &= \mathbf{G} - \mathbf{G}\cdot\left\langle\frac{v_c}{V_0}\mathbf{L}_1\cdot(\mathbf{I}+\mathbf{A}\cdot\mathbf{L}_1)^{-1}\right\rangle\cdot\left[\mathbf{I}+(\mathbf{G}-\mathbf{A}_0)\left\langle\frac{v_c}{V_0}\mathbf{L}_1\cdot(\mathbf{I}+\mathbf{A}\cdot\mathbf{L}_1)^{-1}\right\rangle\right]^{-1}\cdot\mathbf{G};\\ \mathbf{L}_E &= \mathbf{L} + \left\langle\frac{v_c}{V_0}\mathbf{L}_1\cdot(\mathbf{I}+\mathbf{A}\cdot\mathbf{L}_1)^{-1}\right\rangle\cdot\left[\mathbf{I}-\mathbf{A}_0\cdot\left\langle\frac{v_c}{V_0}\mathbf{L}_1\cdot(\mathbf{I}+\mathbf{A}\cdot\mathbf{L}_1)^{-1}\right\rangle\right]^{-1}.\end{aligned} \tag{2.12}$$

The tensor $\mathbf{A}_0$ in (2.11) and (2.12) is determined by the relation (1.4) for $\mathbf{C}=\mathbf{1}$ ($\mathbf{1}$ is the identity two-valued tensor).

A similar result was obtained in [6]. If the inclusions are spherical in shape, then the expressions (2.12) for $\mathbf{L}_E$ and $\mathbf{G}_E$ are identical with those obtained in [5].

B. The case of identical inclusions forming a lattice is now considered. For simplicity, an isotropic base medium and spherical isotropic inclusions are only considered. In this case the relations (2.12) become

$$\begin{aligned}\mathbf{G}_E &= \mathbf{G} - p\mathbf{G}\cdot\mathbf{L}_1\cdot\left(\mathbf{I}+\mathbf{A}\cdot\mathbf{L}_1+\frac{1}{p}\int \mathbf{K}(\mathbf{R})\,S(\mathbf{R})\,dV\cdot\mathbf{L}_1\right)^{-1};\\ \mathbf{L}_E &= \mathbf{L} + p\mathbf{L}_1\cdot\left(\mathbf{I}+\mathbf{A}\cdot\mathbf{L}_1 - p\mathbf{G}\cdot\mathbf{L}_1+\frac{1}{p}\int \mathbf{K}(\mathbf{R})\,S(\mathbf{R})\,dV\cdot\mathbf{L}_1\right)^{-1}.\end{aligned} \tag{2.13}$$

In this case the components of the tensors $\mathbf{K}(\mathbf{R})$ and $\mathbf{A}$ are given by the relations

$$K_{ijkl}(\mathbf{R}) = -\frac{1}{16\pi\mu(1-\nu)R^3}\left[2(2\nu-1)\left(\delta_{ij}\delta_{kl} - 3\delta_{ij}\frac{R_kR_l}{R^2}\right)+\right.$$

$$+\left(\delta_{il}\delta_{jk}-3\delta_{jk}\frac{R_iR_l}{R^2}\right)-\frac{3}{R^2}(\delta_{il}R_jR_k+\delta_{jl}R_iR_k+\delta_{kl}R_iR_j)+15\frac{R_iR_jR_kR_l}{R^4}\Big]_{(ij)(kl)},\quad R=|\mathbf{R}|;$$

$$A_{ijkl}=\frac{1}{3\mu}\left[\delta_{ij}\delta_{kl}-\frac{1}{10(1-\nu)}(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\right],$$

where μ is the shear modulus; ν is the Poisson coefficient for the elastic medium. The function $S(\mathbf{R})$ in (2.13) is given by

$$S(\mathbf{R})=\langle\Theta(\mathbf{r})\Theta(\mathbf{r}')\rangle',$$

where the dash outside the averaging symbol indicates that only those realizations are taken into account for which the points $\mathbf{r}$ and $\mathbf{r}'$ fall within different inclusions. Let $\mathbf{a}^{(1)}$, $\mathbf{a}^{(2)}$, $\mathbf{a}^{(3)}$ be a triple of vectors specifying the elementary Bravais cell of a lattice whose nodes are at the centers of the inclusions. A simple geometrical scrutiny results in the following expression for $S(\mathbf{R})$:

$$S(\mathbf{R})=\sum_{k,m,n=-\infty}^{\infty}{}' J(\mathbf{R}-k\mathbf{a}^{(1)}-m\mathbf{a}^{(2)}-n\mathbf{a}^{(3)}) \tag{2.14}$$

(the dash at the side of the summation symbol indicates the omission of the term $k=m=n=0$) and

$$\frac{1}{p}J(\mathbf{R})=\begin{cases}\left(1-\frac{R}{2r_0}\right)^2\left(1+\frac{R}{12r_0}\right) & \text{for } R\leqslant 2r_0\\ 0 & \text{for } R>2r_0,\end{cases} \tag{2.15}$$

where r_0 is the radius of the inclusions; $J(\mathbf{R})$ is understood in the same sense as in (2.4). By using the regularization of (1.3) and (1.5) one can represent the integral of (2.13) in the form

$$\frac{1}{p}\int\mathbf{K}(\mathbf{R})S(\mathbf{R})\,dV=\frac{1}{p}\int\mathbf{K}(\mathbf{R})[S(\mathbf{R})-p^2]\,dV+p\mathbf{G}=p(\mathbf{G}-\mathbf{A})+\frac{1}{p}\int\mathbf{K}(\mathbf{R})[S(\mathbf{R})-p^2]\,dV,$$

where the last integral is understood as the principal value in the Cauchy sense and converges to $S(\mathbf{R})$ in the (2.14) and (2.15) form. If the inclusions form a cubic lattice, then the integral in (2.13) is equal to

$$\frac{1}{p}\int\mathbf{K}(\mathbf{R})S(\mathbf{R})\,dV=p(\mathbf{G}-\mathbf{A})+\alpha(p)\mathbf{M}. \tag{2.16}$$

The components of the tensor $\mathbf{M}$ are given by

$$\mu(1-\nu)M_{ijkl}=\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-5a^{-4}\sum_{n=1}^{3}a_i^{(n)}a_j^{(n)}a_k^{(n)}a_l^{(n)},\quad a=|\mathbf{a}^{(n)}|,\quad n=1,2,3, \tag{2.17}$$

and the scalar coefficient α is given by the integral

$$\alpha(p)=\frac{1}{16\pi p}\int[S(\mathbf{R})-p^2]\left(\frac{1}{R^3}-5\frac{R_1^2R_2^2+R_1^2R_3^2+R_2^2R_3^2}{R^7}\right)dV,$$

again understood as its principal value in the Cauchy sense.

This integral was numerically evaluated for the function $S(\mathbf{R})$ given by (2.14) and (2.15). The coefficient α against the inclusion density p is shown in Fig. 1.

Substituting (2.16) in (2.13), one finds that the tensors of the elastic properties of a medium which contains a cubic inclusion lattice are of the form

$$\mathbf{G}_E=\mathbf{G}-p\mathbf{G}\cdot\mathbf{L}_1\cdot[\mathbf{I}+p\mathbf{G}\cdot\mathbf{L}_1+(1-p)\mathbf{A}\cdot\mathbf{L}_1+\alpha(p)\mathbf{M}]^{-1}\cdot\mathbf{G};$$

$$\mathbf{L}_E=\mathbf{L}+p\mathbf{L}_1\cdot[\mathbf{I}+(1-p)\mathbf{A}\cdot\mathbf{L}_1+\alpha(p)\mathbf{M}]^{-1}.$$

It is obvious that the symmetry group of these tensors is identical with the symmetry group of a cube, since the latter is the symmetry group of the tensor $\mathbf{M}$ [the expression (2.17)].

§3. The two-dimensional problem is now considered. In this case all constructions are carried out similarly as for the three-dimensional case. For an isotropic medium containing isotropic circular inclusions the tensors of pertinent elastic properties are of the form (2.13). However, the components of the tensors $\mathbf{K}(\mathbf{R})$ and $\mathbf{A}$ are given in the case of two-dimensional stressed state by

$$K_{\alpha\beta\gamma\delta}=\frac{(1+\nu)}{8\pi\mu R^2}\left[\frac{(3-\nu)}{(1+\nu)}\left(\delta_{\alpha\beta}\delta_{\gamma\delta}-2\delta_{\alpha\beta}\frac{R_\gamma R_\delta}{R^2}\right)-\nabla_\alpha\nabla_\beta\left(\frac{R_\gamma R_\delta}{R^2}\right)\right]_{(\alpha\gamma)(\beta\sigma)}; \quad (3.1)$$

$$A_{\alpha\beta\gamma\delta}=\frac{1}{16\mu}\left[8I_{\alpha\beta\gamma\delta}-(1+\nu)(\delta_{\alpha\beta}\delta_{\gamma\delta}+2I_{\alpha\beta\gamma\delta})\right], \quad (3.2)$$

where the Greek subscripts assume the values 1 or 2; **I** is the identity four-valued tensor in the two-dimensional space. The regularization of integrals similar to (1.3), (1.5) are of the same kind as in the three-dimensional case for the tensor **A** in the form of (3.2). Some particular cases are now considered.

I. The Random Field of Inclusions Is Homogenous and Isotropic. By the same considerations as employed for the three-dimensional situation, one obtains the following expressions for the tensors of pertinent elastic properties:

$$\begin{aligned}\mathbf{G}_E &= \mathbf{G}-p\mathbf{G}\cdot\mathbf{L}_1\cdot[\mathbf{I}+p\mathbf{G}\cdot\mathbf{L}_1+(1-p)\mathbf{A}\cdot\mathbf{L}_1]^{-1}\cdot\mathbf{G};\\ \mathbf{L}_E &= \mathbf{L}+p\mathbf{L}_1\cdot[\mathbf{I}+(1-p)\mathbf{A}\cdot\mathbf{L}_1]^{-1}.\end{aligned} \quad (3.3)$$

In the above, the tensor **A** is of the form (3.2).

II. The Inclusions Form a Regular Triangular Lattice. Of course, the symmetry group of the functions $S(\mathbf{R})=\langle\Theta(\mathbf{r})\Theta(\mathbf{r}')\rangle$ ' is in this case identical with the symmetry group of the triangular lattice. The regularization of the integral in(2.13) assumes the form (integration being over the entire plane)

$$\frac{1}{p}\int \mathbf{K}(\mathbf{R})\,S(\mathbf{R})\,d\Omega = p(\mathbf{G}-\mathbf{A})+\frac{1}{p}\int \mathbf{K}(\mathbf{R})\,[S(\mathbf{R})-p^2]\,d\Omega. \quad (3.4)$$

The integral on the right-hand side is understood as the principal value in the Cauchy sense and is convergent at zero as well as at infinity. The symmetry group of this integral must be identical with the symmetry group of the functions $S(\mathbf{R})$. It is known [12] that the basis for four-valued tensors which possess the triangular-lattice symmetry consists of isotropic tensors only. Moreover, in view of the symmetry of the tensor $\mathbf{K}(\mathbf{R})$ with respect to the corresponding subscripts, one can write

$$\int K_{\alpha\beta\gamma\delta}(\mathbf{R})\,[S(\mathbf{R})-p^2]\,d\Omega=\beta\,(\delta_{\alpha\beta}\delta_{\gamma\delta}+2I_{\alpha\beta\gamma\delta}).$$

The coefficient β is obtained by contracting both sides of the above relation with respect to all subscripts,

$$\beta=\frac{1}{8}\int K_{\alpha\alpha\gamma\gamma}(\mathbf{R})\,[S(\mathbf{R})-p^2]\,d\Omega.$$

However, it follows directly from (3.1) that

$$K_{\alpha\alpha\gamma\gamma}(\mathbf{R})=0.$$

In view of the above, the integral on the right-hand side of (3.4) vanishes. By now substituting the regularized value of the integral in (3.4) into (2.13), one finds that in the case of a regular triangular lattice of inclusions the expression for the tensor of pertinent elastic properties is equal to its expression in the case of the inclusion field being homogenous and isotropic as in (3.3).

In Figs. 2 and 3 the diagrams are shown of the relative pertinent elasticity moduli E_E/E and of the shear moduli μ_E/μ of a medium which contains a regular triangular inclusion lattice versus the inclusion concentration for different values of their elasticity parameters; they were computed by using the formulas (3.3) (E_1 is the elas-

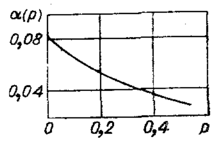

Fig. 1

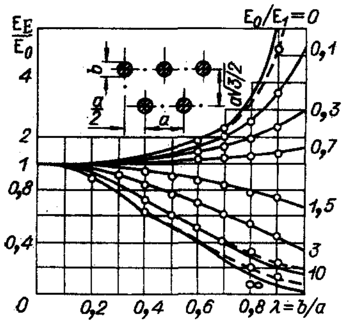

Fig. 2

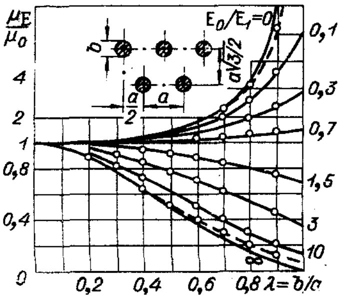

Fig. 3

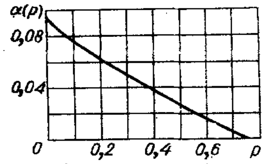

Fig. 4

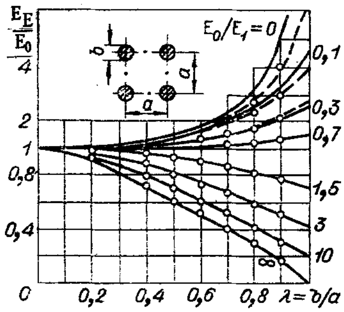

Fig. 5

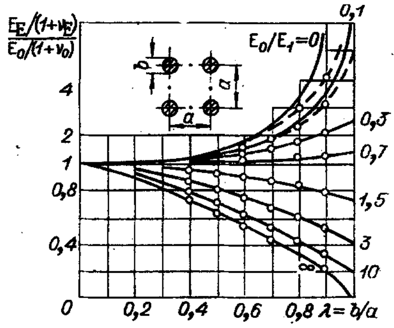

Fig. 6

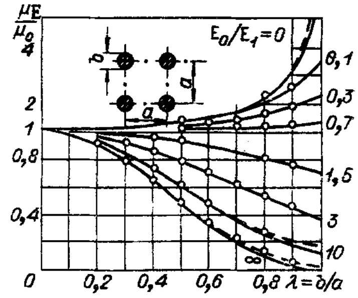

Fig. 7

ticity modulus of the inclusions). The Poisson coefficient of the medium and of the inclusions is $\nu = 0.3$. Also shown are the graphs of these quantities which were evaluated exactly by using the methods developed in [9] for solving biperiodic problems of elasticity theory (solid curves show the exact solution; dashed ones with small circles correspond to the solutions obtained by the self-consistent field methods).

III. The Inclusions Form a Quadratic Lattice. If one regularizes the integral in (2.13) by using (3.4), one can consider the principal value of the integral

$$\frac{1}{p}\int \mathbf{K}(\mathbf{R})\,[S(\mathbf{R}) - p^2]\,d\Omega$$

for this case. It is obvious that the symmetry group of this integral is identical with the symmetry group of the functions S(R) and, consequently, with the symmetry group of a square. By expanding this integral in terms of four-valued tensors of the corresponding symmetry [12] and calculating the coefficients of the expansion, one obtains

$$\frac{1}{p}\int \mathbf{K}(\mathbf{R})\,[S(\mathbf{R}) - p^2]\,d\Omega = \alpha\mathbf{M}. \tag{3.5}$$

The components of the tensor **M** are

$$\frac{\mu}{1+\nu} M_{\alpha\beta\gamma\delta} = \delta_{\alpha\beta}\delta_{\gamma\delta} + 2I_{\alpha\beta\gamma\delta} - 4a^{-4}\sum_{i=1}^{2} a_\alpha^{(i)} a_\beta^{(i)} a_\gamma^{(i)} a_\delta^{(i)}, \tag{3.6}$$

where $\mathbf{a}^{(1)}$ and $\mathbf{a}^{(2)}$ are vectors equal to the side of the square, $a = |\mathbf{a}^{(i)}|$.

The coefficient α is given in an integral form in the principal value sense,

$$\alpha = \frac{1}{8\pi p}\int [S(\mathbf{R}) - p^2]\left(\frac{1}{R^2} - 8\frac{R_1^2 R_2^2}{R^6}\right) d\Omega. \tag{3.7}$$

The expression for the function S(R) is in this case given by

$$S(\mathbf{R}) = \sum_{k,m=-\infty}^{\infty}{}' \, j(\mathbf{R} - k\mathbf{a}^{(1)} - m\mathbf{a}^{(2)}), \tag{3.8}$$

where the dash indicates that the term k = m = 0 is omitted:

$$\frac{1}{p} j(\mathbf{R}) = \begin{cases} \frac{2}{\pi}\left[\operatorname{arctg}\left(\frac{2r_0}{R}\sqrt{1-\frac{R}{2r_0}}\right)\right] - \frac{R}{2r_0}\sqrt{1-\left(\frac{R}{2r_0}\right)^2} & \text{for} \quad R \leqslant 2r_0 \\ 0 & \text{for} \quad R > 2r_0 \end{cases} \tag{3.9}$$

(r_0 is the radius of the inclusions).

The dependence α(p) obtained by numerical integration of (3.7), where S(R) is of the form (3.8) and (3.9), is shown in Fig. 4. By now substituting (3.5) into (3.4) and substituting the result into the expression (2.13), for the tensors of pertinent properties one obtains

$$\begin{aligned} \mathbf{G}_E &= \mathbf{G} - p\mathbf{G}\cdot\mathbf{L}_1\cdot[\mathbf{I} + p\mathbf{G}\cdot\mathbf{L}_1 + (1-p)\ \mathbf{A}\cdot\mathbf{L}_1 + \alpha(p)\mathbf{M}]^{-1}\cdot\mathbf{G}; \\ \mathbf{L}_E &= \mathbf{L} + p\mathbf{L}_1\cdot[\mathbf{I} + (1-p)\mathbf{A}\cdot\mathbf{L}_1 + \alpha(p)\mathbf{M}], \end{aligned} \tag{3.10}$$

where the tensor **A** is of the form (3.2) and the tensor **M** and the coefficient α are given by the relations (3.6) and (3.7).

In Figs. 5-7 the graphs are shown of the relative effective elasticity parameters for a square inclusion lattice evaluated by using the formula (3.10). The exact values given in [9] are also shown (solid curves show the exact solution; dashed ones with circles are solutions obtained by the self-consistent field method).

It can be seen by analyzing these diagrams that for regular structures considered here the solution by the self-consistent field method almost coincides with the exact solution provided the ratio of the medium elasticity modulus to that of the inclusions remains within the limits $0.1 \le E/E_1 \le 10$. For $E/E_1 > 10$ or for $E/E_1 < 0.1$ starting with the inclusion concentrations of $p \geqslant 0.4$ the solutions obtained by the self-consistent field method possess an error which increases with higher concentrations.

Therefore, the assumptions of the self-consistent field method enumerated in Sec. 2 are satisfactorily fulfilled in the analyzed cases. The accuracy in fulfilling these assumptions depends, on the one hand, on the character of the interaction of the inhomogeneities in the elastic medium (in particular, on the interaction constants E/E_1 and p), and on the other hand, on the specific shape of a regular structure. One has reasons to assume that the dependence on the structure is sufficiently "smooth" and that the self-consistent field approximation is, therefore, acceptable for a wide range of different structures, in particular, for a wide "neighborhood" of the situations analyzed here.

It is noticed that in the case of regular structures the assumption that all inclusions are found in the same equivalent field $\hat{\boldsymbol{\varepsilon}}$ is fulfilled exactly. Subsequent improvements are thus possible within the framework of the self-consistent field method by employing a more accurate approximation of the mean field $\boldsymbol{\varepsilon}$.

When the stochastic inclusion fields [7] are investigated, the problem arises of analyzing the part played by the fluctuations of the true stress and strain fields in the neighborhood of each inclusion. It is known [13] that the application of the self-consistent field method to describe phase transitions is the more justified the slower the potential is attenuated due to an individual particle and the smaller the distances between the particles. In particular, the self-consistent field methods produce good results for particles with the Coulomb-type 1/R potential.

In the case of an elastic composite medium, the potential of each separate particle (inclusion) behaves at infinity as 1/R in the two-dimensional case and as $1/R^2$ in the three-dimensional case. It is therefore to be expected that the self-consistent field method for stochastic inclusion fields will prove fully acceptable and more accurate in two-dimensional problems.

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